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SOME RESULTS IN GENERALIZED PARTIAL METRIC SPACES

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Abstract

In this paper we establish a fixed point theorem on generalized partial metric spaces using ϕ continuous and monotone non-decreasing function, and lower semi-continuous function ψ. This result is a modification of Erdal, Wasfi and Kenana's [5] result.
1

1 INTRODUCTION

A generalization of usual metric spaces is a partial metric space. In 1992, Matthews [6, 7] introduced the notion of a partial metric space in which $d(x,x)$ are not necessarily zero. Dhanorkar and Salunke [3] have proved fixed point theorem on partial metric space using continuous and monotonically non-decreasing mapping $\phi, \psi : [0,\infty) \to [0,\infty)$ with $\phi(0) = \psi(0) = 0$.

In 2011, Ahmadi Zand and Dehghan Nezhad [1] introduced generalization of partial metric space. Dhanorkar and Salunke [4] proved a fixed point theorem in partial metric space. Dhanorkar and Salunke [4] proved a fixed point theorem in generalized partial metric spaces with nondecreasing map $\phi : [0,\infty) \to [0,\infty)$ with $\phi(t) < t$, for each $t > 0$. Also Hassen, Erdal and Peyman Salimi[2] proved results on generalized partial metric spaces.

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2 PRELIMINARY

In 2005, Zead Mustafa and Brailey Sims [8] introduced generalized metric space called G-metric space as follows.

Definition 2.1 *Let X be a nonempty set. Suppose a mapping $G : X \times X \times X \to [0,\infty)$ satisfies*

(1) $G(x,y,z) = 0$ *if* $x = y = z$;
(2) $0 < G(x,x,y)$ *for all* $x, y \in X$ *with* $x \neq y$;
(3)$G(x,x,z) \leq G(x,z,y)$ *for all* $x,y,z \in X$ *with* $y \neq z$;
(4)$G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$
(5)$G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ *for all* $x,y,z,a \in X$.

Then (X,G) is called G-metric space.

The definition of generalized partial metric space is given by [1] as follows.

Definition 2.2 *Let X be a nonempty set. Suppose a mapping $g_p : X \times X \times X \to [0,+\infty)$ satisfies*

(1) $0 \leq g_p(x,x,x) \leq g_p(x,x,y) \leq g_p(x,y,z)$ *for all* $x,y,z \in X$;
(2) $g_p(x,y,z) = g_p(z,x,y) = g_p(x,z,y)$ *(Symmetric in all three variables)*;
(3) $g_p(x,y,z) \leq g_p(x,a,a) + g_p(a,y,z) - g_p(a,a,a)$ *for all* $x,y,z,a \in X$;
(4) $x = y = z$ *if* $g_p(x,y,z) = g_p(x,x,x) = g_p(y,y,y) = g_p(z,z,z)\ \forall\ x,y,z \in X$.

Then (X,g_p) is called generalized partial metric space i.e. GP-metric space.

Example 2.1 *Let $X = [0,+\infty)$ and let $g_p : X \times X \times X \to [0,+\infty)$ given by*

$$g_p(x,y,z) = \max\{x,y\} + \max\{y,z\} + \max\{z,x\}.$$

Clearly (X,g_p) is not a G-metric space since $G(x,x,x) = x \neq 0$.

Example 2.2 *Let $X = \{a,b,c\}$ and let $g_p : X \times X \times X \to [0,+\infty)$, defined by $g_p(x,y,z) = 1$,if $x = y = z$;*

$$g_p(a,b,b) = g_p(b,a,a) = 10;$$
$$g_p(a,c,c) = g_p(c,a,a) = 15;$$
$$g_p(b,c,c) = g_p(c,b,b) = 17;$$
$$g_p(a,b,c) = 20.$$

Clearly (X,g_p) is not a G-metric space since $G(a,a,a) = 1 \neq 0$.

Proposition 2.1 *[1] Let (X, g_p) be a GP-metric space and for $x, y, z,$ and $a \in X$ then the following relations are true.*

(i) $g_p(x,y,z) \leq g_p(x,x,y) + g_p(x,x,z) - g_p(x,x,x)$;
(ii) $g_p(x,y,y) \leq 2g_p(x,x,y) - g_p(x,x,x)$;
(iii) $g_p(x,y,z) \leq g_p(x,a,z) + g_p(z,a,a) - g_p(a,a,a)$;
(iv) $g_p(x,y,z) \leq g_p(a,a,x) + g_p(a,a,y) + g_p(a,a,z) - 2g_p(a,a,a)$;
(v) $g_p(x,y,z) + g_p(a,a,a) \leq (2/3)(g_p(x,y,a) + g_p(x,a,z) + g_p(a,y,z))$.

Proposition 2.2 *[1] Every GP-metric space (X, g_p) defines a metric space (X, d_{g_p}) as*

$$d_{g_p}(x,y) = g_p(x,y,y) + g_p(y,x,x) - g_p(x,x,x) - g_p(y,y,y), \text{for all } x, y \in X.$$

Definition 2.3 *[1] (i) A point $x \in X$ in GP-metric space (X, g_p) is said to be the limit of the sequence $\{x_n\}$ or $x_n \to x$, if*

$$\lim_{n,m\to\infty} g_p(x, x_n, x_m) = g_p(x,x,x).$$

In this case, we say that the sequence $\{x_n\}$ is GP-convergent to x.
(ii)A sequence $\{x_n\}$ in GP-metric space (X, g_p) is said to be Cauchy iff

$$\lim_{n,m,l\to\infty} g_p(x_n, x_m, x_l) \text{ is finite.}$$

(iii)A GP-metric space (X, g_p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that

$$\lim_{n,m,l\to\infty} g_p(x_n, x_m, x_l) = g_p(x,x,x).$$

Proposition 2.3 *Let (X, g_p) GP-metric space. Then for the sequence following statements are equivalent*

(i) $\{x_n\}$ convergent to x,
(ii) $g_p(x_n, x_n, x) \to g_p(x,x,x)$ as $n \to \infty$,
(iii) $g_p(x_n, x, x) \to g_p(x,x,x)$ as $n \to \infty$

for all $x, y \in X$.

Proposition 2.4 *Let (X, g_p) complete GP-metric space. Then*

(i) If $g_p(x,y,z) = 0$, then $x = y = z$,
(ii) If $x = y = z = 0$, then $g_p(x,y,z) = 0$,
(ii) If $x \neq y \neq z \neq 0$, then $g_p(x,y,z) > 0$.

Lemma 2.1 *Assume $x_n \to z$ as $n \to \infty$ in a GP-Metric Space (X, g_p) such that $g_p(z,z,z) = 0$. Then, $\lim_{n\to\infty} g_p(x_n, y, y) = g_p(z,y,y)$ for every $y \in X$.*

Proof: Since $\lim_{n\to 0} g_p(x_n, z, z) = g_p(z,z,z) = 0$. We have

$$\begin{aligned} g_p(x_n,y,y) &\leq g_p(x_n,y,z) + g_p(z,z,y) - g_p(z,z,z), \\ &= g_p(x_n,y,z) + g_p(z,z,y). \\ \text{Also } g_p(z,y,y) &\leq g_p(z,y,x_n) + g_p(x_n,x_n,y) - g_p(x_n,x_n,x_n), \\ &= g_p(z,y,x_n) + g_p(x_n,x_n,y). \end{aligned}$$

So, we get

$$\lim_{n\to\infty} (g_p(x_n,y,y) - g_p(z,y,y)) = 0.$$

Hence

$$\lim_{n\to\infty} g_p(x_n,y,y) = g_p(z,y,y).$$

3 MAIN RESULTS

In this section we prove our main result which gives us conditions for existence and uniqueness of a fixed point for a certain type of functions defined on GP- metric spaces.

Theorem 3.1 *Let (X, g_p) be a complete GP-metric space and $T : X \to X$ be a mapping satisfying*

$$\psi(g_p(Tx,Ty,Tz)) \leq \psi(M(x,y,z)) - \phi(M(x,y,z)) \tag{3.1}$$

for all $x, y, z \in X$, where

$$M(x,y,z) = \max\left\{g_p(x,y,z), g_p(y,Ty,Tz)\frac{1 + g_p(x,Tx,Tx)}{1 + g_p(x,y,z)}\right\}, \tag{3.2}$$

$\psi : [0,\infty) \to [0,\infty)$ is a continuous and monotone non-decreasing function with $\psi(t) = 0$ if and only if $t = 0$ and $\phi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Proof: Let x_0 be an arbitrary point in X and sequence $\{x_n\}$ in X such that

$$Tx_n = x_{n+1} \text{ i.e. } T^n x_0 = x_n, n = 1,2,3,\ldots \quad (3.$$

If there exists n such that $x_n = x_{n+1}$ then x_n is a fixed point of T . Now, suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Now

$$\begin{aligned}\psi(g_p(x_n, x_{n+1}, x_{n+1})) &= \psi(g_p(Tx_{n-1}, Tx_n, Tx_n)) \\ &\leq \psi(M(x_{n-1}, x_n, x_n)) - \phi(M(x_{n-1}, x_n, x_n)) \qquad (3.4)\end{aligned}$$

where

$$M(x_{n-1}, x_n, x_n) = \max\left\{g_p(x_{n-1}, x_n, x_n), g_p(x_n, x_{n+1}, x_{n+1})\frac{1+g_p(x_{n-1}, x_n, x_n)}{1+g_p(x_{n-1}, x_n, x_n)}\right\}. \qquad (3.5)$$

Hence, we get

$$\begin{aligned}\psi(g_p(x_n, x_{n+1}, x_{n+1})) \leq \psi\Big(\max\Big\{g_p(x_{n-1}, x_n, x_n), g_p(x_n, x_{n+1}, x_{n+1}\Big\}\Big)- \\ \phi\Big(\max\Big\{g_p(x_{n-1}, x_n, x_n), g_p(x_n, x_{n+1}, x_{n+1}\Big\}\Big). \qquad (3.6)\end{aligned}$$

If $g_p(x_{n-1}, x_n, x_n)) \leq g_p(x_n, x_{n+1}, x_{n+1})$ then equation (3.6)becomes

$$\begin{aligned}\psi(g_p(x_n, x_{n+1}, x_{n+1}) &\leq \psi(g_p(x_n, x_{n+1}, x_{n+1}) - \phi(g_p(x_n, x_{n+1}, x_{n+1})), \qquad (3.7)\\ &< \psi(g_p(x_n, x_{n+1}, x_{n+1})\end{aligned}$$

which is a contradiction. Hence we must have

$$g_p(x_n, x_{n+1}, x_{n+1}) \leq g_p(x_{n-1}, x_n, x_n),$$

$$\psi(g_p(x_n, x_{n+1}, x_{n+1}) \leq \psi(g_p(x_{n-1}, x_n, x_n) - \phi(g_p(x_{n-1}, x_n, x_n)). \qquad (3.8)$$

Therefore $\{x_n\}$ be a non-increasing sequence of positive real numbers. Thus there exists $L \geq 0$ Such that

$$\lim_{n\to\infty} g_p(x_n, x_{n+1}, x_{n+1}) = L.$$

Now if $L > 0$, then taking the upper limit in (3.8) as $n \to \infty$, we get

$$\begin{aligned}\psi(L) &\leq \psi(L) - \lim_{n\to\infty} \phi(g_p(x_{n-1}, x_n, x_n)), \qquad (3.9)\\ &= \psi(L) - \phi(L),\\ &< \psi(L) \qquad (3.10)\end{aligned}$$

which is a contradiction. Therefore

$$\lim_{n\to\infty} g_p(x_n, x_{n+1}, x_{n+1}) = 0. \qquad (3.11)$$

Now, we have to show that sequence $\{x_n\}$ is g_p -Cauchy. Therefore consider subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$, there exist $\epsilon > 0$ and n_k is the smallest index for which $n_k > m_k > \epsilon$ such that $g_p(x_n, x_m, x_m) \geq \epsilon$ this means $g_p(x_{n-1}, x_m, x_m) < \epsilon$. We can write

$$\begin{aligned}\epsilon \leq g_p(x_{n_k}, x_{m_k}, x_{m_k}) &\leq g_p(x_{n_k}, x_{n_k-1}, x_{n_k-1}) + g_p(x_{n_k-1}, x_{m_k}, x_{m_k})\\ &\quad - g_p(x_{n_k-1}, x_{n_k-1}, x_{n_k-1})\\ &\leq \epsilon + g_p(x_{n_k}, x_{n_k-1}, x_{n_k-1}) \qquad (3.12)\end{aligned}$$

as $k \to \infty$ we get

$$g_p(x_{n_k}, x_{m_k}, x_{m_k}) = \epsilon. \qquad (3.13)$$

Now

$$\begin{aligned}g_p(x_{n_k}, x_{m_k}, x_{m_k}) &= g_p(x_{n_k}, x_{n_k-1}, x_{n_k-1}) + g_p(x_{n_k-1}, x_{m_k}, x_{m_k})\\ &\quad - g_p(x_{n_k-1}, x_{n_k-1}, x_{n_k-1})\\ &\leq g_p(x_{n_k}, x_{n_k-1}, x_{n_k-1}) + g_p(x_{n_k-1}, x_{m_k}, x_{m_k})\\ &\leq g_p(x_{n_k}, x_{n_k-1}, x_{n_k-1}) + g_p(x_{n_k-1}, x_{m_k-1}, x_{m_k-1})\\ &\quad + g_p(x_{m_k-1}, x_{m_k}, x_{m_k}) - g_p(x_{m_k-1}, x_{m_k-1}, x_{m_k-1})\\ &\leq g_p(x_{n_k}, x_{n_k-1}, x_{n_k-1}) + g_p(x_{n_k-1}, x_{m_k-1}, x_{m_k-1})\\ &\quad + g_p(x_{m_k-1}, x_{m_k}, x_{m_k}) \qquad (3.14)\end{aligned}$$

and

$$\begin{aligned}g_p(x_{n_k-1}, x_{m_k-1}, x_{m_k-1}) &= g_p(x_{n_k-1}, x_{n_k}, x_{n_k})+\\ &\qquad g_p(x_{n_k}, x_{m_k-1}, x_{m_k-1}) - g_p(x_{n_k}, x_{n_k}, x_{n_k})\\ &\leq g_p(x_{n_k-1}, x_{n_k}, x_{n_k}) + g_p(x_{n_k}, x_{m_k-1}, x_{m_k-1})\\ &\leq g_p(x_{n_k-1}, x_{n_k}, x_{n_k}) + g_p(x_{n_k}, x_{m_k}, x_{m_k})+\\ &\qquad g_p(x_{m_k}, x_{m_k-1}, x_{m_k-1}) - g_p(x_{m_k}, x_{m_k}, x_{m_k})\\ &\leq g_p(x_{n_k-1}, x_{n_k}, x_{n_k}) + g_p(x_{n_k}, x_{m_k}, x_{m_k})+\\ &\qquad g_p(x_{m_k}, x_{m_k-1}, x_{m_k-1}). \qquad (3.15)\end{aligned}$$

Taking $k \to \infty$ in the above two inequalities we get

$$\lim_{n\to\infty} g_p(x_{n_k-1}, x_{m_k-1}, x_{m_k-1}) = \epsilon. \qquad (3.16)$$

We can write equation (3.1) as

$$\begin{aligned}\psi(g_p(x_{m_k}, x_{n_k}, x_{n_k})) &= \psi(g_p(Tx_{m_k-1}, Tx_{n_k-1}, Tx_{n_k}))\\ &\leq \psi(M(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}))-\\ &\qquad \phi(M(x_{m_k-1}, x_{n_k-1}, x_{n_k-1})), \qquad (3.17)\end{aligned}$$

for all $x, y, z \in X$, where

$$M(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}) = \max\Big\{g_p(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}), \tag{3.18}$$

$$g_p(x_{n_k-1}, Tx_{n_k-1}, Tx_{n_k-1})\frac{1 + g_p(x_{m_k-1}, Tx_{m_k-1}, Tx_{m_k-1})}{1 + g_p(x_{m_k-1}, x_{n_k-1}, x_{n_k-1})}\Big\} \tag{3.19}$$

Using (3.11),(3.13) and (3.16), we get

$$\lim_{k\to\infty} M(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}) = \epsilon \tag{3.20}$$

Now equation (3.17) becomes

$$\begin{aligned}\psi(\epsilon) &\le \psi(\epsilon) - \lim_{k\to\infty}\inf \phi(M(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}))\\ &\le \psi(\epsilon) - \phi(\epsilon)\\ &< \psi(\epsilon)\end{aligned}$$

which is a contradiction. So, we have

$$g_p(x_{n_k}, x_{m_k}, x_{m_k}) = 0 \tag{3.21}$$

Hence $\{x_n\}$ is g_p-Cauchy in (X, g_p). Since (X, g_p) is a complete partial metric space then there exists $x \in X$ such that $\lim_{n\to\infty} g_p(x_n, x, x) = g_p(x, x, x)$. Since $\lim_{n\to\infty} g_p(x_n, x_m, x_m) = 0$ by lemma (2), we get $g_p(x, x, x) = 0$. Now our aim to prove x is fixed point of T. Suppose $Tx \neq x$, we can write

$$\begin{aligned}\psi(g_p(x_n, Tx, Tx)) &= \psi(g_p(Tx_{n-1}, Tx, Tx))\\ &\le \psi(M(x_{n-1}, x, x)) - \phi(M(x_{n-1}, x, x))\end{aligned}$$

where

$$M(x_{n-1}, x, x) = \max\Big\{g_p(x_{n-1}, x, x), g_p(x, Tx, Tx)\frac{1 + g_p(x_{n-1}, x_n, x_n)}{1 + g_p(x_{n-1}, x, x)}\Big\} \tag{3.22}$$

as $n \to \infty$, we get

$$\begin{aligned}\psi(g_p(x, Tx, Tx)) &= \psi(g_p(x, Tx, Tx))\\ &\le \psi(g_p(x, Tx, Tx)) - \phi(g_p(x, Tx, Tx))\\ &< \psi(g_p(x, Tx, Tx))\end{aligned}$$

which is not true, hence $Tx = x$, x is fixed point of T.
Uniqueness: Suppose y is another point such that $Ty = y$ and $y \neq x$.

$$\begin{aligned}\psi(g_p(x, y, y)) &= \psi(g_p(Tx, Ty, Ty))\\ &\le \psi(M(x, y, y)) - \phi(M(x, y, y))\\ &\le \psi(g_p(x, y, y)) - \phi(g_p(x, y, y))\\ &< \psi(g_p(x, y, y))\end{aligned}$$

which is contradiction. Hence $x = y$.

In above Theorem 3.1, taking $\psi(t) = t$ for all $t \in [0, \infty)$ and $\phi(t) = (1-k)t$ for all $t \in [0, \infty)$ with $k \in (0, 1)$ we get following result

Corollary 3.1 *Let (X, g_p) be a complete GP-metric space, and $T : X \to X$ be a mapping satisfying*

$$g_p(Tx, Ty, Tz) \le k \max\Big\{g_p(x, y, z), g_p(y, Ty, Tz)\frac{1 + g_p(x, Tx, Tx)}{1 + g_p(x, y, z)}\Big\} \tag{3.23}$$

for all $x, y, z \in X$, where $k \in (0, 1)$. Then T has a unique fixed point.

Example 3.1 *Let $X = [0, 1]$ and let $g_p(x, y, z) = \max\{x, y, z\}$, where $x, y, z \in X$, then (X, g_p) is generalized partial metric space such that $g_p(x, y, z) = \max\{x, y, z\}$ Suppose $T : X \to X$ such that $Tx = \frac{x^2}{1+x}$, for all $x \in X$ and $\phi, \psi : [0, \infty) \to [0, \infty)$ such that $\phi(t) = \frac{t}{1+t}$ and $\psi(t) = t$. Without loss of generality, assume that $x \ge y \ge z$*

$$g_p(Tx, Ty, Tz) = \max\Big\{\frac{x^2}{1+x}, \frac{y^2}{1+y}, \frac{z^2}{1+z}\Big\} = \frac{x^2}{1+x}$$

and

$$M(x, y, z) = \max\Big\{g_p(x, y, z), g_p(y, Ty, Tz)\frac{1 + g_p(x, Tx, Tx)}{1 + g_p(x, y, z)}\Big\} = x$$

we get

$$g_p(Tx, Ty, Tz) = \max\Big\{\frac{x^2}{1+x}, \frac{y^2}{1+y}, \frac{z^2}{1+z}\Big\} = \frac{x^2}{1+x} \le x - \frac{x}{1+x} = \frac{x^2}{1+x}$$

Thus, it satisfies all conditions of Theorem 3.1. T has a fixed point, i.e. $x = 0$ is the required point.

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References

[1] **M.R.Ahmadi Zand,A.Dehghan Nezhad,** A generalization of partial metric space, J. Contemporary Appl. Math.,1(1),(2011),86-93.

[2] **Hassen Aydi, Erdal Karapnar and Peyman Salimi**, Some fixed point results in GP-metric spaces,J.Appl. Math., Vol.2012, Article ID 891713, 16 pages doi:10.1155/2012/891713.